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INERTIAL MOTION OF AN ABSOLUTELY RIGID BODY ON A TWO-DEGREES-OF-FREEDOM JOINT†

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Equations of motion are derived for an absolutely rigid body attached to a fixed base by a two-degrees-of-freedom joint. The behaviour of this system is investigated in the special case when there are no applied forces other than the reaction forces in the joint. Possible types of motion that may occur are determined, depending on the relations among the components of the inertia tensor of the body and the first integrals of the system (the kinetic energy and projection of the angular momentum onto the fixed axis of the joint). The qualitative and quantitative characteristics of the motion are found.

Papers on the dynamics of an absolutely rigid body with a fixed point generally assume that the mechanical system in question has three degrees of freedom. This is the situation, in particular, when the body is attached to a fixed base by a ball-and-socket joint. In engineering systems, however, one often encounters rigid bodies attached to a base by a two-degrees-of-freedom joint, consisting of a fixed axis and a movable one, which are usually mutually perpendicular. Such systems have two degrees of freedom, but the set of kinematically possible motions is still quite rich. In this paper we shall consider the motion of such a system by inertia, i.e. when there are no external forces other than the reaction at the joint. The inertial motion of other mechanical systems with two degrees of freedom has been studied by various researchers. The present paper uses techniques similar to those of [1, 2], which were devoted to the inertial motion of a plane articulated linkage of two rigid bodies when the axis of one of the joints is fixed in the inertial system of coordinates.

1. EQUATION OF MOTION

Consider an absolutely rigid body attached to a fixed base by a two-degrees-of-freedom joint with mutually perpendicular axes (Fig. 1). The joint is assumed to be ideal, i.e. the friction in its axes is disregarded. The motion will be described in terms of two orthogonal Cartesian systems of coordinates: a fixed (inertial) system $X_1X_2X_3$ and a system $x_1x_2x_3$ rigidly attached to the rigid body. The poles of both coordinate systems are at the point of intersection O of the joint axes; the X_3 and x_1 axes point along the fixed and moving axes of the joint, respectively. All the kinematically possible positions of the body (i.e. of the moving system of coordinates $x_1x_2x_3$) relative to the fixed system of coordinates $X_1X_2X_3$ may be described in terms of two angles: the angle α between the X_1 and x_1 axes, and the angle β between the x_2 axis and the X_1X_2 plane. The angles α and β, which will be taken as generalized coordinates, may be treated as the angles of two successive rotations through which one can transfer the rigid body from its initial position ($\alpha=\beta=0$, the moving system of coordinates coincides with the fixed

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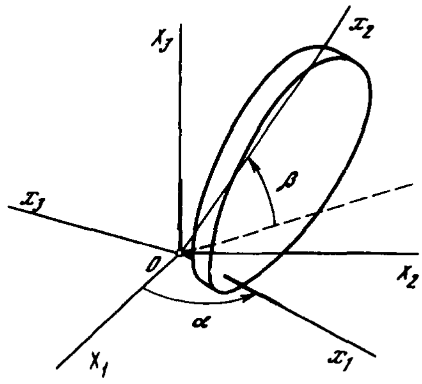

Fig. 1.

one) to the present position. The first rotation (through the angle α) takes place about the X_3 axis (the fixed axis of the joint) and is described by the matrix

$$\Gamma_\alpha = \begin{Vmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{Vmatrix}$$

The second rotation (through the angle β) takes place about the x_1 axis (the moving axis of the joint) and corresponds to the matrix

$$\Gamma_\beta = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{Vmatrix}$$

The transformation matrix from $X_1X_2X_3$ coordinates to $x_1x_2x_3$ coordinates is the product $\Gamma = \Gamma_\beta\Gamma_\alpha$.

Let ω_i denote the projection of the angular velocity vector ω of the body onto the x_i ($i=1$, 2, 3) axis. The kinematic equations expressing the components ω_i in terms of the generalized coordinates α and β and the velocities $\dot\alpha$ and $\dot\beta$ are

$$\omega_1 = \dot\beta, \quad \omega_2 = \dot\alpha\sin\beta, \quad \omega_3 = \dot\alpha\cos\beta \tag{1.1}$$

Equations (1.1) may be derived, in particular, directly from the matrix Γ, as follows (see, e.g. [3])

$$\omega_1 = (\Gamma\dot\Gamma')_{32}, \quad \omega_2 = (\Gamma\dot\Gamma')_{13}, \quad \omega_3 = (\Gamma\dot\Gamma')_{21}$$

where $(\Gamma\dot\Gamma')_{ij}$ are the appropriate elements of the matrix $\Gamma\dot\Gamma'$, the prime denotes transposition and the dot denotes differentiation with respect to time.

The kinetic energy of the motion of a rigid body with a fixed point is

$$T = \tfrac{1}{2}(J\boldsymbol{\omega}, \boldsymbol{\omega}), \quad \boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) \tag{1.2}$$

where J is the inertia tensor of the body relative to the fixed point. Expanding the scalar product in (1.2) taking (1.1) into account, we obtain

$$T = \tfrac{1}{2}K(\beta)\dot{\alpha}^2 + \tfrac{1}{2}J_{11}\dot{\beta}^2 - b(\beta)\dot{\alpha}\dot{\beta} \tag{1.3}$$
$$K(\beta) = J_{22}\sin^2\beta + J_{33}\cos^2\beta - 2J_{23}\sin\beta\cos\beta$$
$$b(\beta) = J_{12}\sin\beta + J_{13}\cos\beta$$

where J_{ii} $(i=1, 2, 3)$ are the axial moments of inertia and $J_{ij}=J_{ji}$ $(i\neq j, i, j=1, 2, 3)$ are the products of inertia of the body in $x_1x_2x_3$ coordinates. We shall assume that the inertia ellipsoid of the body is non-degenerate and the inertia tensor J is positive definite.

The Lagrange equations corresponding to (1.3) are

$$\begin{aligned} &K(\beta)\ddot{\alpha} - b(\beta)\ddot{\beta} + [(J_{22}-J_{33})\sin 2\beta - 2J_{23}\cos 2\beta]\dot{\alpha}\dot{\beta} - (J_{12}\cos\beta - J_{13}\sin\beta)\dot{\beta}^2 = Q_\alpha \\ &-b(\beta)\ddot{\alpha} + J_{11}\ddot{\beta} - \tfrac{1}{2}[(J_{22}-J_{33})\sin 2\beta - 2J_{23}\cos 2\beta]\dot{\alpha}^2 = Q_\beta \end{aligned} \tag{1.4}$$

where Q_α and Q_β are the generalized forces corresponding to the generalized coordinates α and β, respectively. In physical terms, Q_α and Q_β are the torques about the X_3 and x_1 axes, respectively, of the active forces applied to the rigid body.

2. INERTIAL MOTION ($Q_\alpha = Q_\beta = 0$)

In this case the system of equations (1.4) has two first integrals: the kinetic energy T (1.3) and the quantity

$$L = \partial T / \partial\dot{\alpha} = K(\beta)\dot{\alpha} - b(\beta)\dot{\beta} \tag{2.1}$$

This fact may be deduced from general theorems of mechanics, without appealing to the Lagrange equations. Conservation of kinetic energy follows from the fact that the mechanical system in question is holonomic and scleronomous, while the generalized forces acting on it are zero. Conservation of the quantity L (2.1) is a consequence of the fact that the generalized coordinate α is cyclic. The number L is the projection of the angular momentum $J\omega$ of the system onto the fixed axis X_3.

Solving (3.1) for $\dot{\alpha}$ and substituting the resulting expression into (2.3), we have

$$a(\beta)\dot{\beta}^2/2 + \Pi(\beta, L) = T \tag{2.2}$$
$$a(\beta) = [(J_{11}J_{22} - J_{12}^2)\sin^2\beta + (J_{11}J_{33} - J_{13}^2)\cos^2\beta -$$
$$-2(J_{11}J_{23} + J_{12}J_{13})\sin\beta\cos\beta]/K(\beta), \quad \Pi(\beta, L) = L^2/(2K(\beta))$$
$$a > 0, \quad K > 0$$

The inequalities $a>0$, $K>0$ follow from the fact that the inertia tensor of the body is positive definite.

Indeed, a direct check shows that the expressions for a and K may be written as scalar products

$$a = (J\mathbf{u}, \mathbf{u}), \quad K = (J\mathbf{v}, \mathbf{v})$$
$$\mathbf{u} = \|1, b(\beta)\sin\beta / K(\beta), \; b(\beta)\cos\beta / K(\beta)\|, \quad \mathbf{v} = \|0, \sin\beta, \cos\beta\|$$

which are positive because the inertia tensor J is positive definite. That a and K are positive may also be deduced from the fact that $\Pi = L^2/(2K)$ is the kinetic energy of the body when $\dot{\beta}=0$, and $a\dot{\beta}^2/2$ is the kinetic energy when $L=0$. Since the kinetic energy of a body with a non-degenerate inertia ellipsoid is positive for $\omega\neq 0$, it is clear that $a>0$ and $K>0$.

Equation (2.2) describes the variation of the angle β and reduces to a quadrature

$$\int_{\beta_0}^{\beta}\left\{\frac{a(\xi)}{2[T-\Pi(\xi,L)]}\right\}^{1/2} d\xi = \pm(t-t_0) \tag{2.3}$$

where $\beta_0 = \beta(t_0)$ and t_0 is a suitably chosen initial time. The sign on the right-hand side of (2.3) is the same as that of $\dot\beta$ if $\dot\beta \neq 0$ or with the sign of the expression $\partial\Pi/\partial\beta$ if $\dot\beta = 0$. Note that the integral on the left-hand side of (2.3) may be expressed in terms of elementary and elliptic functions.

From a geometrical standpoint Eq. (2.2) describes the projection of the phase trajectory of system (1.4) for $Q_\alpha = Q_\beta = 0$ onto the phase plane β, $\dot\beta$ for the generalized coordinate β. It follows from (2.2) that the motion as a function of the angular variable β is analogous to the motion of a conservative mechanical system with one degree of freedom, with kinetic energy $a(\beta)\dot\beta^2/2$ and potential energy $\Pi(\beta, L)$. The constant T is then an analogue of the total mechanical energy of a system with one degree of freedom. Qualitatively, therefore, motion as a function of β may be investigated by phase-plane methods, as when constructing phase trajectories for a one-degree-of-freedom conservative system.

We shall first consider the reduced potential energy $\Pi(\beta, L)$ as a function of β in the interval $0 \leqslant \beta \leqslant 2\pi$. Transforming to the doubled argument 2β expression (2.2) for Π reduces to the form

$$\Pi(\beta,L) = L^2/(2K),\quad 2K = J_{22} + J_{33} + R\cos(2\beta+\nu) \tag{2.4}$$

$$\cos\nu = (J_{33}-J_{22})/R,\quad \sin\nu = 2J_{23}/R$$

$$R = [(J_{33}-J_{22})^2 + 4J_{23}^2]^{1/2}$$

Let us assume that the equalities $J_{33} = J_{22}$, $J_{23} = 0$ do not hold simultaneously. It follows from (2.4) that the function $\Pi(\beta, L)$, $L \neq 0$, has four extremum points in the range $0 \leqslant \beta \leqslant 2\pi$, say $\beta = \beta_i$ $(i = 1, 2, 3, 4)$ and these are precisely the extremum points of the function $\cos(2\beta+\nu)$

$$\beta_i = \{-\arccos[(J_{33}-J_{22})/R] + \pi i\}/2,\quad J_{23} > 0 \tag{2.5}$$

$$\beta_i = \{\arccos[(J_{33}-J_{22})/R] + \pi(i-1)\}/2,\quad J_{23} < 0$$

$$\beta_i = \pi(i-1)/2,\quad J_{23} = 0;\quad i = 1,2,3,4$$

If $J_{23} > 0$ $(J_{23} < 0)$, then $\Pi(\beta, L)$ reaches a maximum (minimum) at β_1 and β_3 and a minimum (maximum) at β_2 and β_4. If $J_{23} = 0$ and $J_{22} > J_{33}$ $(J_{22}J_{33})$, then β_1 and β_3 are maximum (minimum) points, while β_2 and β_4 are minimum (maximum) points of $\Pi(\beta, L)$. At these extremum points

$$\Pi(\beta_i, L) = L^2(J_{22} + J_{33} \mp R)^{-1} \tag{2.6}$$

where R has the minus sign for a maximum and the plus sign for a minimum. The function $\Pi(\beta, L)$ takes the same value at all maximum (minimum) points; we shall denote this value henceforth by Π_{max} (Π_{min}).

It follows from (2.2) that $T - \Pi(\beta, L) \geqslant 0$ and so, in view of (2.6), the first integrals T and L satisfy the relationship $T \geqslant \Pi_{min}(L^2)$.

Figure 2 illustrates the construction of phase trajectories in the β, $\dot\beta$ plane and plots a few phase curves for different relations between T and L^2. Curve 1 corresponds to $T > \Pi_{max}$, curve 2 to $T = \Pi_{max}$ and curve 3 to $\Pi_{min} < T < \Pi_{max}$.

If $\Pi_{min} < T < \Pi_{max}$, the motion of the body as a function of β consists of periodic oscillations about the points β_i (see (2.5)) where the reduced potential energy $\Pi(\beta, L)$ has a minimum. The period τ_υ of the oscillations equals double the value of the integral on the left-hand side of (2.3), if one puts $\beta_0 = \beta_-$, $\beta = \beta_+$, where β_- and β_+ are two consecutive values of the angle β at

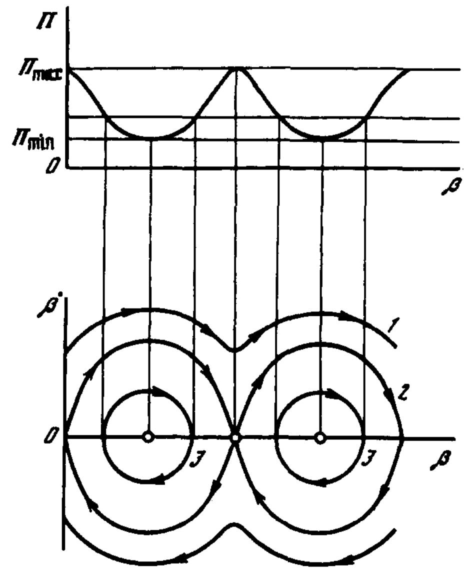

Fig. 2.

which the angular velocity $\dot{\beta}$ vanishes and between which there is a minimum point of $\Pi(\beta, L)$

$$\tau_{\upsilon} = 2\int_{\beta_-}^{\beta_+} \left\{ \frac{a(\beta)}{2[T - \Pi(\beta, L)]} \right\}^{1/2} d\beta, \quad \beta_- < \beta_i < \beta_+ \tag{2.7}$$

The values of β_- and β_+ are found from the equation $\Pi(\beta, L) - T = 0$, where $\Pi(\beta, L)$ is determined from (2.4). Solving this equation we obtain

$$\beta_{\mp} = \beta_i \mp A, \quad A = \frac{1}{2}\arccos\frac{L^2 - T(J_{22} + J_{33})}{TR} \tag{2.8}$$

whence it follows that the amplitude of the oscillations is A.

If $T > \Pi_{max}$, then β varies monotonically, that is, the body rotates about the fixed x_1 axis. This rotation is periodic with period

$$\tau_r = \int_0^{2\pi} \left\{ \frac{a(\beta)}{2[T - \Pi(\beta, L)]} \right\}^{1/2} d\beta \tag{2.9}$$

If $T = \Pi_{max}$, the representative point of the system describes a separatrix, but if $T = \Pi_{min}$ the phase trajectories shrink to a point ($\beta = \beta_i$, $\dot{\beta} = 0$), i.e. the body moves at a fixed angle $\beta = \beta_i$, at which the function $\Pi(\beta, L)$ has a minimum. When $T = \Pi_{max}$ the motion may again correspond to a fixed value of $\beta = \beta_i$ at which $\Pi(\beta, L)$ has a maximum. As follows from (2.1), if $\beta = \beta_i =$ const, the angle α will vary at a constant rate

$$\dot{\alpha} = 2\Pi / L = 2T / L = 2L(J_{22} + J_{33} \mp R)^{-1} \tag{2.10}$$

This means that the body will rotate uniformly at angular velocity (2.10) about the fixed X_3 axis, while the moving x_2 axis will make an angle β with the X_1X_2 plane. It follows from Routh's theory (see also Fig. 2) that if the potential energy at β_i is $\Pi(\beta_i, L) = \Pi_{min}$, the motion is stable with respect to β and $\dot{\beta}$. but if $\Pi(\beta_i, L) = \Pi_{max}$, the motion in unstable.

Note that the quantity $K = K(\beta)$ defined in (1.3) and occurring in formulae (2.2) and (2.4) for the function $\Pi(\beta, L)$ is the moment of inertia of the rigid body about the fixed X_3 axis, if the moving x_2 axis makes an angle β with the X_1X_2 coordinate plane. Indeed, the moment of inertia I_e of the body about an arbitrary axis (with unit vector $\mathbf{e}$) passing through the point O may be expressed in terms of the inertia tensor J as

$$I_e = (J\mathbf{e}, \mathbf{e}) \tag{2.11}$$

To calculate I_e one must express the inertia tensor J and unit vector $\mathbf{e}$ in terms of the same system of coordinates. In our case $\mathbf{e}$ is a unit vector along the X_3 axis, which has the following components in the moving frame $x_1x_2x_3$

$$\mathbf{e} = (0, \sin\beta, \cos\beta) \tag{2.12}$$

Substituting (2.12) into (2.11), we obtain $I_e = K(\beta)$. Accordingly, minimizing (maximizing) the function $\Pi(\beta, L)$ in (2.2) is equivalent to maximizing (minimizing) the moment of inertia of the body relative to the X_3 axis.

It follows from the foregoing that, when there are no active forces, a rigid body attached to a base by a two-degrees-of-freedom joint may rotate uniformly about a fixed (X_3) axis only if its moment of inertia about the axis is either a maximum or a minimum. In the case of a maximum moment of inertia the motion is stable; for a minimum, it is unstable.

Remark. We have assumed that the equalities $J_{22} = J_{33}$, $J_{23} = 0$ do not hold simultaneously. If they do, the expressions for $a(\beta)$, $\Pi(\beta, L)$ and $K(\beta)$ in (2.2) become much simpler

$$a(\beta) = [J_{11}J_{22} - b^2(\beta)]/J_{22}, \quad \Pi(\beta, L) = L^2/(2J_{22}), \quad K(\beta) = J_{22}$$

The functions $\Pi(\beta, L)$ and $K(\beta)$ are constants, independent of β. We recall that $K(\beta)$ is the moment of inertia of the body about the fixed X_3 axis; the fact that this quantity is constant implies that the moment of inertia remains unchanged in all possible configurations of the system. In the special case considered, there are only two qualitatively distinct types of motion. If $T - \Pi = T - L^2/(2J_{22}) > 0$, the body performs periodic rotations with respect to β with the period defined by (2.9). If $T - L^2/(2J_{22}) = 0$ any constant values of β (and only constant values) satisfy the differential equation (2.2). In that case the body rotates uniformly about the X_3 axis at angular velocity $\dot{\alpha} = L/J_{22}$. Unlike the general case, rotation is possible for any inclination of the moving x_2 axis to the X_1X_2 plane. Oscillatory motion as a function of β is impossible when $J_{22} = J_{33}$, $J_{23} = 0$.

We will now investigate the motion as a function of α. solving Eq. (2.1) for $\dot{\alpha}$, we obtain

$$\dot{\alpha} = L/K(\beta) + b(\beta)\dot{\beta}/K(\beta) \tag{2.13}$$

where $K(\beta)$ is determined from (1.3) or (2.4). We note here that $K(\beta)$ is a π-periodic function.

In principle, one could find the angle α from Eq. (2.13) as an explicit function of time by using (2.3). Namely, one uses (2.3) to express β as a function of time, inserts $\beta(t)$ into (2.13) and integrates both sides of (2.13) with respect to time from t_0 (the initial instant of time) to t (the present instant of time). Generally speaking, however, this can be done only by numerical means, as the integrals—involved in particular, the integral (2.3)—cannot be expressed in terms of functions that can be investigated by analytical techniques. We will therefore confine ourselves to a qualitative study of the motion as a function of α.

We will first consider the case when $T > \Pi_{\max}$, when the system performs periodic rotations as a function of β, with period τ_r (see (2.9)). We first prove the following lemma.

Lemma. Let $f(x)$ be an integral π-periodic function. Then for any x

$$\int_x^{x+2\pi} f(\xi)\sin\xi d\xi = 0, \qquad \int_x^{x+2\pi} f(\xi)\cos\xi d\xi = 0 \tag{2.14}$$

We shall prove the first of these equalities; the proof of the other is analogous. Express the integral (2.14) as a sum, bisecting the range of integration at the point $x+\pi$. In the interval $[x+\pi,\ x+2\pi]$ we use the change of variables $\xi=\eta+\pi$ and use the properties of the functions $f(\eta)$ and $\sin\eta$ to obtain

$$\int_{x+\pi}^{x+2\pi} f(\xi)\sin\xi d\xi = -\int_{x}^{x+\pi} f(\xi)\sin\xi d\xi$$

which implies the first equality of (2.14).

Integrate Eq. (2.13) with respect to t from t_0 to $t_0+n\tau_r$, where τ_r is the period of rotation with respect to β and n an arbitrary natural number. Using the equalities (see (2.3))

$$\dot{\beta}dt = d\beta,\quad dt = \pm\{a(\beta)/[2(T-\Pi(\beta,L))]\}^{1/2}d\beta \tag{2.15}$$

we obtain, after reduction

$$\alpha(t_0+n\tau_r)-\alpha(t_0) = n[L\Omega(T,L)\pm B] \tag{2.16}$$

$$\Omega(T,L) = \int_0^{2\pi} F(T,L,\beta)d\beta,\quad B = \int_0^{2\pi}\frac{b(\beta)}{K(\beta)}d\beta$$

$$F(T,L,\beta) = \frac{1}{K(\beta)}\left[\frac{K(\beta)J_{11}-b^2(\beta)}{2K(\beta)T-L^2}\right]^{1/2}$$

The double sign ($\pm$) in (2.15) and (2.16) reflects the fact that the sense of the body's rotation about the moving axis x_1 may vary. The positive sense (counterclockwise as observed from the end of the unit vector of the x_1 axis) is indicated by the plus sign and the negative sense by the minus sign.

It follows from the lemma proved above that $B=0$. Hence, if $L\neq 0$, then when $T>\Pi_{max}$ (rotation with respect to β) each revolution of the body about the x_1 axis changes the angle α by a systematic increment equal in absolute value to $|L|\Omega(T,\ L)$. The direction of the increment depends on the sign of the constant L. Thus, the magnitude of the angle α may become arbitrarily large as time passes. It should be noted that, generally speaking, the variation of $\alpha(t)$ is not monotonic.

If natural numbers m and n exist such that

$$n\,|\,L\,|\Omega = 2\pi m \tag{2.17}$$

then the motion of the body as a whole is periodic (in the sense that the body will return to its initial state although the angles α and β are defined only up to multiples of 2π) and the least period is $\tau=n\tau_r$, where n is the least natural number n satisfying (2.17). In other words, the motion of the system will be periodic if the number $\mu=|L|\Omega(T,\ L)/(2\pi)$ is rational, in which case the minimum period will be $\tau=n\tau_r$, where n is the least natural denominator of the fraction μ. If μ is irrational, the motion is non-periodic.

In the special case $L=0$ the motion with respect to α is periodic with least period $\tau=\tau_r$, and, unlike the general case $L\neq 0$, it is bounded (the coordinate α cannot take arbitrarily large values). In that case one can state that when $T=\Pi_{max}$ the motion with respect to α is rotational (though not in general monotonic) if $L\neq 0$ and oscillatory if $L=0$.

Let us now consider the case in which $\Pi_{min}<T<\Pi_{max}$ and the system performs oscillations with respect to β with period τ_υ and amplitude A (see (2.7) and (2.8)). Integrating Eq. (2.13) with respect to time from t_0 to $t_0+n\tau_\upsilon$, where n is a natural number, we obtain

$$\alpha(t_0+n\tau_\upsilon)-\alpha(t_0) = L\Omega_1(L,T)n,\quad \Omega_1(L,T) = 2\int_{\beta_-}^{\beta_+} F(T,L,\beta)d\beta \tag{2.18}$$

where β_- and β_+ are the minimum and maximum of the oscillating angle β, defined by (2.8). Note that the integral with respect to t of the second term on the right-hand side of (2.13), from t_0 to $t_0+n\tau_\upsilon$ vanishes, since after replacing the variable of integration in accordance with (2.15) the integral, considered over each oscillation period separately, reduces to the sum of two integrals of the function $b(\beta)/K(\beta)$, one of which is evaluated from β_- to β_+ and the other from β_+ to β_-. Equation (2.18) is the analogue of (2.16) in the case of oscillations with respect to β.

Analysing Eq. (2.18), as done previously for (2.16), we reach the following conclusions. Since oscillations with respect to β are only possible when $L\neq 0$, it follows that α may take arbitrarily large values. If the quotient $\nu=|L\Omega_1|/(2\pi)$ is a rational number, the body as a whole will move periodically, the least period being $\tau=n\tau_\upsilon$, where n is the least natural denominator of the fraction ν. If ν is irrational the motion is non-periodic.

Finally, we shall establish simple sufficient conditions for the rotation with respect to α to be monotonic, i.e. sufficient conditions for the generalized velocity $\dot\alpha$ to have a fixed sign. Since $K(\beta)>0$, we deduce from (2.13) that $\dot\alpha$ does not change sign if $|L|>|b(\beta)\dot\beta|$ or, equivalently

$$L^2>b^2(\beta)\dot\beta^2 \tag{2.19}$$

Solving (2.2) for $\dot\beta^2$ and substituting the result into (2.19), we obtain

$$L^2J_{11}>2Tb^2(\beta) \tag{2.20}$$

which is equivalent to (2.19). Since $b^2(\beta)=(J_{12}\sin\beta+J_{13}\cos\beta)^2\leqslant J_{12}^2+J_{13}^2$, it follows that inequality (2.20) will certainly hold if

$$L^2J_{11}>2T(J_{12}^2+J_{13}^2) \tag{2.21}$$

Hence inequality (2.21) is a sufficient condition for rotation with respect to α to be monotonic.

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